# SOME INVARIANTS AND COVARIANTS OF TERNARY COLLINEATIONS

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A DISSERTATION

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.
1907

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## Introduction.

1. The analytical basis of the present paper is the form of Grassmann's Lückenausdruck which Gibbs called a dyadic. This, as the sequel shows, is merely a general bilinear function from which the variables are omitted. It may then represent a collineation or correlation and may be manipulated practically like the ordinary symbolical bilinear form.

Starting with this as a basis, the object is in the next place to give an interpretation by means of the invariant theory of various double products suggested by Gibbs and incidentally to obtain some of the properties of the invariants and covariants involved. The field of operation is plane projective geometry and the products are formed according to the combinatory multiplication of Grassmann.

Finally, in the third part, there is considered a skew symmetric function of any number of collineations which is called an alternant. It is a combinant, linear in the coefficients of each collineation, and presenting in some ways for functions of two sets of variables properties analogous to those of the expressions resulting from the combinatory multiplication of linear manifolds.

## Part I. Notation.

### I. *The open product or dyadic.*

2. In a space of two dimensions a sum of mixed products of similar construction, each containing a single factor $x$, may be written in the form

$$A_1x \cdot B_1 + A_2x \cdot B_2 + A_3x \cdot B_3,$$

where the dot is used to show that the order of multiplication is from left to right. $A_i$, $B_i$ and $x$ are geometric quantities, points or lines of the plane, and all products are formed according to the combinatory multiplication. This may be considered as resulting from the operation of $x$ on the expression

$$A_1(\ ) \cdot B_1 + A_2(\ ) \cdot B_2 + A_3(\ ) \cdot B_3,$$

the operation consisting in placing the variable $x$ in the parentheses. This last expression is an example of what Grassman called an *open* product.*

* "Ausdehnungslehre" (1878), p. 265.

Gibbs wrote the open product in the form

$$A_1B_1 + A_2B_2 + A_3B_3$$

and from the nature of its construction called it a *dyadic*.* The variable is supposed to operate on the dyadic from the outside and so give as result

$$xA_1 \cdot B_1 + xA_2 \cdot B_2 + xA_3 \cdot B_3$$

or

$$A_1 \cdot B_1x + A_2 \cdot B_2x + A_3 \cdot B_3x$$

according as $x$ is used as prefactor or postfactor.

In the present paper the notation of Gibbs will be used and combinatory products will be represented either by placing the letters in parentheses or by placing a bar over them. It is found convenient to use the parentheses when the product reduces to a scalar, or number, and in all other cases to use the bar. Unless otherwise expressly stated the variable will enter the dyadic as postfactor, i. e., the dyadic will operate on the variable. From analogy with the ordinary symbolism for a row product we shall write

$$AB = A_1B_1 + A_2B_2 + A_3B_3.$$

It is to be observed that $A_i$ and $B_i$ in this expression have a definite size or intensity. If they are only projectively given the dyadic will have the form

$$AB = \lambda_1 A_1B_1 + \lambda_2 A_2B_2 + \lambda_3 A_3B_3,$$

where the $\lambda$'s are numbers determined when definite intensities are given to $A_i$ and $B_i$.

3. As an operator the dyadic gives a linear transformation of quantities contragredient to $B_i$. For, $x$ being such a quantity, since $(B_ix)$ is a number,

$$A(Bx) = \lambda_1(B_1x)A_1 + \lambda_2(B_2x)A_2 + \lambda_3(B_3x)A_3$$

which as a function of $A_i$ is a simple manifold involving $x$ linearly.

There are two cases of present interest. When $A_i$ and $B_i$ are contragredient we have a collineation; when cogredient, a correlation.

A dyadic of the form

$$a\alpha = \lambda_1 a_1\alpha_1 + \lambda_2 a_2\alpha_2 + \lambda_3 a_3\alpha_3,$$

where the $a$'s are points and the $\alpha$'s lines, represents a point collineation.† In

* GIBBS's "Vector Analysis" (Wilson), chap. V.

† In the notation of CLEBSCH this is of course

$$(a\xi)(\alpha x) = \Sigma\lambda_i(a_i\xi)(\alpha_i x) = 0,$$

when $x$ is given and $\xi$ variable. If $\xi$ were given and $x$ variable the dyadic would be written $\alpha a$. The dyadic is thus regarded not as giving a connex, but as setting up a definite transformation.

particular to the point $\overline{\alpha_2\alpha_3}$ corresponds the point

$$a(a\alpha_2\alpha_3) = \lambda_1(\alpha_1\alpha_2\alpha_3)a_1.$$

To the vertices of the triangle $\alpha$ then correspond the points $a_i$. Since the triangle $\alpha$ merely presents a set of points to be operated upon it is obvious that this may be chosen at random, the collineation then determining $a$ as its correspondent. While $a\alpha$ as an operator gives the collineation of points it involves internally the collineation of triads.

Similarly the dyadic

$$\alpha\beta = \lambda_1\alpha_1\beta_1 + \lambda_2\alpha_2\beta_2 + \lambda_3\alpha_3\beta_3$$

represents a correlation in which the lines $\alpha_i$ correspond to the points of the triangle $\beta$. A like interpretation may be given for the dual cases $aa$ and $ab$.

## II. *Tetrad and counter-tetrad.*

4. We have seen that the dyadic in trinomial form involves the correspondence of triads. Since, however, a collineation or correlation in the plane is determined by four pairs of corresponding elements, it is of greater interest to have the dyadic involve a correspondence of sets of four. And as a dyadic always operates on a contragredient quantity this end can only be attained by the use of a self dual scheme of four-point and four-line.

With a 4-point we associate the 4-line obtained by taking the polar of each point with respect to the triangle of the other three. It is well known then that conversely the 4-point is obtained by taking the polar of each line with respect to the triangle of the other three. These mutually related systems have been called *tetrad* and *counter-tetrad.**

Supposing the points $a_i$ to satisfy only one linear relation (which is the only case of interest) their intensities may be chosen such that

(1) $$a_1 + a_2 + a_3 + a_4 = 0.$$

Operating on this identity with the products $\overline{a_i a_j}$ we find that the triple products $(a_i a_j a_k)$ are in absolute value all equal. If then we write

$$(a_2 a_3 a_4) = 4,$$

we obtain

(2) $$(a_i a_j a_k) = \pm 4 \qquad (i<j<k),$$

where the sign is positive or negative according as $a_i a_j a_k$ is complementary to an odd or an even term in the sequence $a_1 a_2 a_3 a_4$.

* F. MORLEY, *Trans. Amer. Math. Society*, vol. 4, p. 291.

Making use of these formulas the counter-tetrad $\alpha_i$ may be written in the canonical form

$$(3)\qquad \begin{aligned} 4\alpha_1 &= -\overline{a_2 a_3} - \overline{a_3 a_4} - \overline{a_4 a_2}, \\ 4\alpha_2 &= \phantom{-}\overline{a_4 a_1} + \overline{a_1 a_3} + \overline{a_3 a_4}, \\ 4\alpha_3 &= -\overline{a_1 a_2} - \overline{a_2 a_4} - \overline{a_4 a_1}, \\ 4\alpha_4 &= \phantom{-}\overline{a_1 a_2} + \overline{a_2 a_3} + \overline{a_3 a_1}. \end{aligned}$$

From these equations by addition we obtain

$$(4)\qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0.$$

Multiplying the equations (3) by $a_i$ and making use of (2) it is easily seen that

$$(5)\qquad (a_i\alpha_i) = 3, \qquad (a_i\alpha_j) = -1 \qquad (i \neq j).$$

We have here sixteen equations. From the identity

$$\sum_i (a_i\alpha_j) = \sum_j (a_i\alpha_j) = 0$$

it is seen however that seven of these equations are superfluous, their effect being to make $a_i$ and $\alpha_i$ subject to conditions (1) and (4). When one tetrad is given there are then nine conditions to be satisfied by the other. And since a tetrad subject to the conditions (1) or (4) in addition to its eight geometrical constants involves an undetermined intensity it follows that there is a single solution. The equations (5) may therefore be taken as canonically defining a tetrad and counter-tetrad. Their symmetry in $a_i$ and $\alpha_i$ indicates the mutuality previously mentioned.

From the equations (3) by direct multiplication we obtain

$$(6)\qquad \overline{\alpha_i \alpha_j} = a_l - a_k,$$

where $i, j, k, l$ is a positive permutation of the numbers 1, 2, 3, 4. Multiplying by $\alpha_k$ and making use of (5)

$$(\alpha_i\alpha_j\alpha_k) = \pm 4 \qquad (i<j<k),$$

the sign being positive or negative according as $\alpha_i\alpha_j\alpha_k$ is an odd or even minor of $\alpha_1\alpha_2\alpha_3\alpha_4$. From the symmetry of the entire system in $a_i$ and $\alpha_i$ we may finally write

$$(8)\qquad \overline{a_i a_j} = \alpha_l - \alpha_k,$$

where the rule of subscripts is the same as before.

5. The application of the preceding to the study of dyadics in the plane is now simple. Consider the collineation

$$(9)\qquad \sum (a_2 a_3 a_4)(\beta_2\beta_3\beta_4) a_1 \beta_1.$$

Taking $a_i$ and $\beta_i$ subject to the conditions (2) and (7) the products $(a_i a_j a_k)(\beta_i \beta_j \beta_k)$ all become equal and (9) becomes

$$a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + a_4\beta_4.$$

Operating on this with $b_1$, a point of the counter-tetrad of $\beta$, and making use of (5) we get

$$3a_1 - a_2 - a_3 - a_4 = 4a_1.$$

The points $b_i$ pass by (9) into the points $a_i$. The collineation therefore transforms the associated system $b$, $\beta$ into the system $a$, $\alpha$ and so the dyadic in this form involves a correspondence of tetrads.

In the same way we see that

$$(10) \qquad \sum(\alpha_2\alpha_3\alpha_4)(\beta_2\beta_3\beta_4)\alpha_1\beta_1$$

is a correlation which transforms the counter-tetrad of $\beta$ into $\alpha$ and so carries the system $b$, $\beta$ into the system $\alpha$, $a$.

## Part II. Multiple Products.

### I. *Multiple products are complete invariants.*

6. With two dyadics $AA'$, $BB'$ is connected a form $\overline{AB}\,\overline{A'B'}$ which Gibbs called the double product of the two dyadics.* It is formed by multiplying the dyadics distributively, each pair of terms combining to form a product in which the prefactor is product of prefactors and postfactor product of postfactors. Gibbs showed that this double multiplication is distributive with respect to a resolution of either dyadic or is invariantive as is readily seen upon expansion.

So with a system of dyadics are a series of multiple products given by the various ways in which prefactors and postfactors may be independently combined. From their construction it is evident that such forms retain their significance when the prefactors and postfactors are transformed separately and therefore belong to the class of functions that Pasch called *complete* invariants.† If the dyadics appear as transformations operating on the elements of a certain field, since a transformation of postfactors amounts to a transformation of that field, it follows that the geometric interpretation of a multiple product must involve an arbitrary initial field. If, for example, a system of collineations and correlations in the plane operate upon four points, the multiple products will give results independent of the initial 4-point, i. e., invariants and covariants

* Loc. cit., p. 306.

The function here considered is a double product only in the sense that it is formed by a certain double process. It is neither the scalar nor the vector but the combinatorial double product. All of these have certain properties in common which characterize double multiplication.

† "Vollkommene Invariante," *Math. Ann.*, Bd. 52, p. 128.

of the resulting tetrads. Illustrations of this property will appear in the discussions that follow.

## II. *Apolarity of collineations.*

7. Consider two contragredient collineations $a\alpha$ and $\beta b$, the first an operator on points, the second an operator on lines. They have a double product invariant $(a\beta)(\alpha b)$. When this vanishes the collineations will be called *apolar*.

To see the meaning of this write

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$
$$\beta b = \mu_1 \beta_1 b_1 + \mu_2 \beta_2 b_2 + \mu_3 \beta_3 b_3,$$

and take $\Delta b = \Delta\alpha$ as reference triangle. That is, place

$$(b_i \alpha_i) = 1, \qquad (b_i\, \alpha_k) = 0.$$

We then have

$$(a\beta)(\alpha b) = \lambda_1 \mu_1 (a_1 \beta_1) + \lambda_2 \mu_2 (a_2 \beta_2) + \lambda_3 \mu_3 (a_3 \beta_3).$$

This obviously vanishes when for each value of the subscript $a_i$ is on $\beta_i$. Two triangles so related that each point of the one lies on the corresponding line of the other will be called *incident*. Now $\Delta a$ and $\Delta\beta$ are the correspondents of the reference triangle with respect to $a\alpha$ and $\beta b$. Hence apolarity is the condition under which two collineations can send a triangle into a pair of incident triangles.*

A collineation apolar to the identical collineation sends certain triangles into incident or inscribed triangles. Such a collineation has been called normal.

Write the given collineation

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3$$

and the identical collineation

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,$$

where $\Delta a = \Delta\alpha$ is reference triangle. The apolarity condition is then

$$\lambda_1 (a_1 \alpha_1) + \lambda_2 (a_2 \alpha_2) + \lambda_3 (a_3 \alpha_3) = (a\alpha) = 0.$$

Hence the condition for normal collineation is the vanishing of what Gibbs called the *scalar*† of the dyadic.

8. Another interpretation for apolarity is obtained by using the system of counter-tetrads explained in Art. 4. The invariant $(a\beta)(\alpha b)$ gives the condition that the collineation

$$a(\alpha b)\beta$$

* M. PASCH, *Math. Annalen*, Bd. 23, p. 431.

† Loc. cit., p. 275.

be normal. Hence if $a\alpha$ and $\beta b$ operating on a certain tetrad and counter-tetrad give a 4-point $a_i$ and a 4-line $\beta_i$, the collineation which transforms $b_i$ (the counter-tetrad of $\beta_i$) into $a_i$ will be normal. According to (9) the collineation which carries $b_i$ into $a_i$ is

$$\sum (a_2 a_3 a_4)(\beta_2 \beta_3 \beta_4) a_1 \beta_1 .$$

Therefore the condition that the collineations $a\alpha$ and $\beta b$ be apolar is

$$\sum (a_2 a_3 a_4)(\beta_2 \beta_3 \beta_4)(a_1 \beta_1) = 0, \tag{11}$$

where $a_i$ and $\beta_i$ correspond respectively through $a\alpha$ and $\beta b$ to a tetrad of points and its counter-tetrad of lines.

It is to be observed that (11) is linear in each of the quantities $a_i$ and $\beta_i$. Hence if all of those quantities except a point $a_k$ are given it will lie on a definite line, and if all except a line $\beta_k$ are given it will pass through a definite point. Therefore a 4-point and 4-line subject to the condition (11) determine a tetrad of lines through $a_i$ and a tetrad of points on $\beta_i$, consisting, in fact, of the evectants of (11) with respect to $a_i$ and $\beta_i$.

If we write $a_i$ and $\beta_i$ in such form that the equations (2) and (7) hold, the apolarity condition (11) may be written

$$(a_2 a_3 a_4)(a_1 \beta_1) - (a_1 a_3 a_4)(a_2 \beta_2) + (a_1 a_2 a_4)(a_3 \beta_3) - (a_1 a_2 a_3)(a_4 \beta_4) = 0.$$

Placing $(a_2 a_3 a_4) = 4$, we get for the evectant with respect to $a_1$

$$4\gamma_1 = 4\beta_1 - (a_2 \beta_2)\overline{a_3 a_4} + (a_3 \beta_3)\overline{a_2 a_4} - (a_4 \beta_4)\overline{a_2 a_3}.$$

Placing $\alpha_i$ as counter-tetrad of $a_i$ this may be written in the form

$$4\gamma_1 = 4\left\{ \beta_1 - \tfrac{1}{4}\left[ (a_1 \beta_1)\alpha_1 + (a_2 \beta_2)\alpha_2 + (a_3 \beta_3)\alpha_3 + (a_4 \beta_4)\alpha_4 \right] \right\},$$

as is readily seen upon multiplying the right hand members of both equations by each of the points $a_i$ and using the identity

$$(a_1 \beta_1) + (a_2 \beta_2) + (a_3 \beta_3) + (a_4 \beta_4) = 0,$$

to which (11) reduces. Since the expression in brackets is symmetrical with respect to the numbers 1, 2, 3, 4 we may finally write

$$\gamma_i = \beta_i - \eta \tag{12}$$

where

$$\eta = \tfrac{1}{4}\left[ (a_1 \beta_1)\alpha_1 + (a_2 \beta_2)\alpha_2 + (a_3 \beta_3)\alpha_3 + (a_4 \beta_4)\alpha_4 \right].$$

From (12) and (8) we have

$$\gamma_2 - \gamma_1 = \beta_2 - \beta_1 = \overline{b_3 b_4},$$

where $b$ is the counter-tetrad of $\beta$. Hence if we order to the lines $\gamma_i$ the

points $b_i$ it follows that each pair of lines intersect on the join of the remaining pair of points. Such a 4-point and 4-line may be called *chiastic.**

From the symmetry of (11) in $\alpha_i$ and $\beta_i$ we are now able to write the evectant with respect to $\beta_i$ in the form

(13) $$c_i = a_i - y$$

where

$$y = \tfrac{1}{2}\,[(a_1\beta_1)b_1 + (a_2\beta_2)b_2 + (a_3\beta_3)b_3 + (a_4\beta_4)b_4].$$

Therefore the four-point $c_i$ and the four-line $\alpha_i$ are chiastic.

The geometric interpretation of apolarity then leads to the following statement of a theorem of PASCH:†

Two apolar collineations transform any four-point and associated four-line into a four-point $a_i$ and a four-line $\beta_i$ such that there is a four-line through $a_i$ chiastic to the counter-tetrad of $\beta_i$ and a four-point on $\beta_i$ chiastic to the counter-tetrad of $a_i$.

From (13) it is observed that the tetrads $c_i$ and $a_i$ are perspective, $y$ being the center of perspective. Since counter-tetrads are chiastic this is a special case of a theorem of PASCH which states that any pair of four-points chiastic to the same four-line are perspective.

9. It has already been observed that the apolarity of the collineations $a\alpha$ and $\beta b$ amounts to the vanishing of the linear invariant (scalar) of

$$a(\alpha b)\beta.$$

For brevity we shall write

$$s_1 = a\alpha, \qquad s_2 = b\beta,$$

$$\sigma_1 = \alpha a, \qquad \sigma_2 = \beta b.$$

And generally we shall designate a collineation by $s$ and its inverse by $\sigma$. The collineation written above is then $s_1 s_2$. As a convenient abbreviation we shall denote the linear invariant of any collineation by the symbol of that collineation placed in parentheses. Thus the apolarity of $a\alpha$ and $\beta b$ is given symbolically by

$$(s_1 s_2) = 0.‡$$

From the definition it follows immediately that

$$(s_1 s_2) = (s_2 s_1) = (\sigma_1\sigma_2) = (\sigma_2\sigma_1).$$

Similarly we shall sometimes find it convenient to denote the linear invariant

* Cf. Sir ROBERT BALL, "Theory of Screws," p. 306.

† *Math. Annalen*, Bd. 26, p. 311.

‡ $s_1$ and $s_2$ in this case will be called harmonic, retaining the word apolar to express the relation of $s_1$ and $\sigma_2$, or $s_2$ and $\sigma_1$.

of the product of any number of collineations $s_1, s, \cdots, s_r$ by $(s_1 s_2 \cdots s_r)$. Writing the collineations in the form $a\alpha$, $b\beta$, $c\gamma$, etc., it is immediately seen that

(14) $$(s_1 \cdots s_{r-1} s_r) = (s_r s_1 \cdots s_{r-1}).$$

That is, the linear invariant of the product of any number of collineations is not affected by a cyclic permutation of those collineations.

10. Suppose we have three collineations each of which transforms a 3-line $\alpha_i$ into a 3-line $\beta_i$. If $a_i$ and $b_i$ are the points of the triangles $\alpha$ and $\beta$ the collineations may be written

$$\sigma_1 = \lambda_1 \beta_1 a_1 + \lambda_2 \beta_2 a_2 + \lambda_3 \beta_3 a_3,$$
$$\sigma_2 = \mu_1 \beta_1 a_1 + \mu_2 \beta_2 a_2 + \mu_3 \beta_3 a_3,$$
$$\sigma_3 = \nu_1 \beta_1 a_1 + \nu_2 \beta_2 a_2 + \nu_3 \beta_3 a_3,$$

where $\lambda_i$, $\mu_i$, and $\nu_i$ are all different from zero. Furthermore let

$$s = \rho_1 c_1 \alpha_1 + \rho_2 c_2 \alpha_2 + \rho_3 c_3 \alpha_3$$

be a non-singular collineation apolar to $\sigma_1$, $\sigma_2$, and $\sigma_3$. Taking $\alpha$ as reference triangle the conditions required are

$$\rho_1 \lambda_1 (\beta_1 c_1) + \rho_2 \lambda_2 (\beta_2 c_2) + \rho_3 \lambda_3 (\beta_3 c_3) = 0,$$
$$\rho_1 \mu_1 (\beta_1 c_1) + \rho_2 \mu_2 (\beta_2 c_2) + \rho_3 \mu_3 (\beta_3 c_3) = 0,$$
$$\rho_1 \nu_1 (\beta_1 c_1) + \rho_2 \nu_2 (\beta_2 c_2) + \rho_3 \nu_3 (\beta_3 c_3) = 0.$$

These equations can only coexist when either

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = 0$$

or

$$\rho_1 (\beta_1 c_1) = \rho_2 (\beta_2 c_2) = \rho_1 (\beta_3 c_3) = 0.$$

The first of these expresses that the collineations are linearly related; the second that the 3-line $\beta_i$ and the 3-point $c_i$ are incident. Hence if a non-singular collineation is apolar to three linearly independent collineations having a common triangle pair, it transforms the first of those triangles into one incident to the second. In particular, if $\alpha_i$ and $\beta_i$ are identical, their triangle is transformed by $s$ into an inscribed triangle and is consequently a Pasch triangle* of $s$.

Suppose a second set of collineations $T_1$, $T_2$ and $T_3$ which transform a 3-point

* F. MORLEY, loc. cit., p. 295.

$d_i$ into a 3-point $c_i$. The same is then true of any collineation of the net

$$\rho_1 T_1 + \rho_2 T_2 + \rho_3 T_3.$$

Suppose one of these collineations should transform $a_i$ into a 3-point incident to $\beta_i$. By the last paragraph the conditions required are

$$\rho_1(S_1 T_1) + \rho_2(S_1 T_2) + \rho_3(S_1 T_3) = 0,$$
$$\rho_1(S_2 T_1) + \rho_2(S_2 T_2) + \rho_3(S_2 T_3) = 0,$$
$$\rho_1(S_3 T_1) + \rho_2(S_3 T_2) + \rho_3(S_3 T_3) = 0.$$

These equations may be satisfied if

$$(15) \qquad \begin{vmatrix} (S_1 T_1) & (S_1 T_2) & (S_1 T_3) \\ (S_2 T_1) & (S_2 T_2) & (S_2 T_3) \\ (S_3 T_1) & (S_3 T_2) & (S_3 T_3) \end{vmatrix} = 0.$$

From the symmetry of this condition in $S$ and $T$ we conclude that if there is a collineation which transforms $d_i$ into $c_i$ and $a_i$ into a triad incident to $\beta_i$, then there is a collineation that transforms $b_i$ into $a_i$ and $c_i$ into a triad incident to $\delta_i$.

If in the above theorem we take $a_i$ equal to $b_i$ and $c_i$ equal to $d_i$ we have the theorem of Hun that the relation of Pasch triangle and fixed triangle in a normal collineation is mutual.

## III. *The intermediate.**

11. Take in the next place two cogredient point collineations $a\alpha$ and $b\beta$. They have a double product $\overline{ab}\,\overline{\alpha\beta}$.† This is a covariant line collineation which has been called the intermediate of $a\alpha$ and $b\beta$.

The geometric interpretation is easily seen. Let $\eta$ (Fig. 1) be the correspondent of any line $\xi$ two of whose points are $x$ and $y$. Then by a well known identity

$$(16) \qquad \eta = \overline{ab}(\alpha\beta\xi) = \overline{ab}\{(\alpha x)(\beta y) - (\alpha y)(\beta x)\} = \overline{x'y''} - \overline{x''y'}$$

in which $x'$ and $x''$, $y'$ and $y''$ are the correspondents of $x$ and $y$ with respect to $a\alpha$ and $b\beta$. Therefore $\eta$ is a line through the join of $\overline{x'y''}$ and $\overline{x''y'}$, or, as we may say, through the cross-join of the correspondents of $x$ and $y$. Since $x$ and $y$ are any points on $\xi$, *$\eta$ is the locus of cross-joins of pairs of points, on $\xi$*. This from the known construction of a polarity amounts to saying that $\overline{x'x''}$ envelopes a conic tangent to $\overline{x'y'}$ and $\overline{x''y''}$ at their junction with $\eta$.

* A. B. Coble, *Trans. Amer. Math. Soc.*, vol. 4, p. 70. Prof. Morley has used the word Clebschian to represent a form of this kind (*Trans.*, vol. 4, p. 471).

† In the symbolic notation of Clebsch this would of course be written

$$(abx)(\alpha\beta\xi) = 0,$$

where $\xi$ is given and $x$ variable.

In case of three points $x$, $y$, $z$ of $\xi$ the preceding construction involves Pascal's theorem for the hexagon inscribed in a two-line.

In case the two collineations are the same, $\eta$ is the join of $x'$ and $y'$ and the intermediate reduces to the reciprocal or line form. Hence the line equation

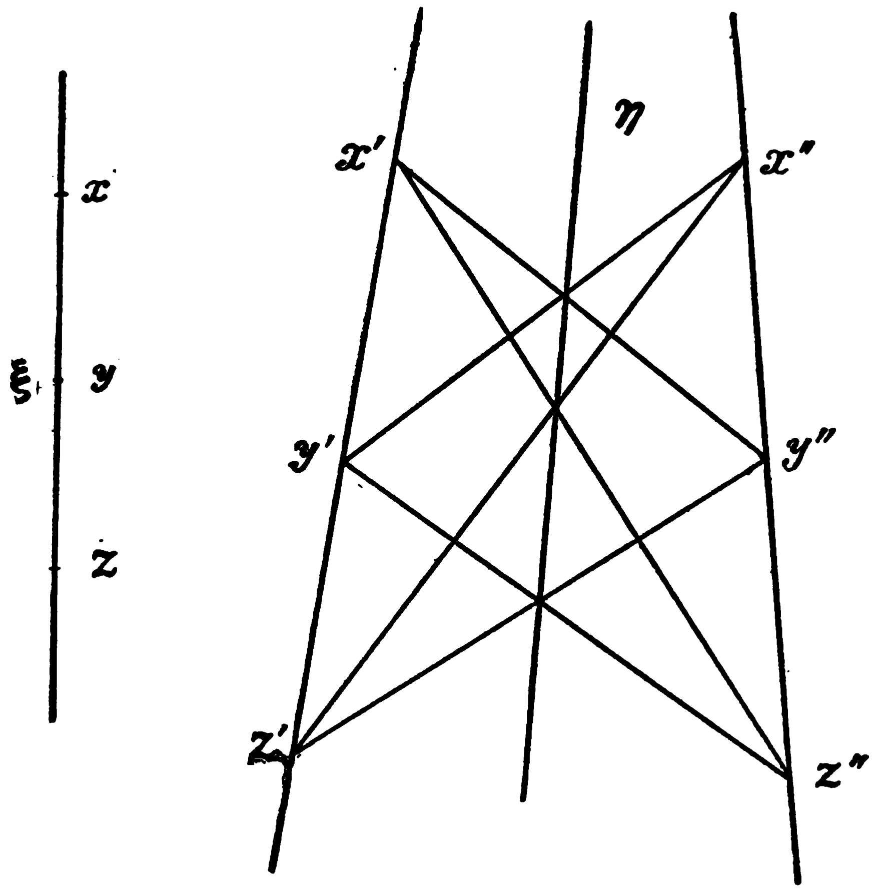


FIG. 1.

of a given collineation is gotten by taking the double product of the dyadic with itself.

12. An identity for which we shall find frequent use is obtained by developing $(abx)(\alpha\beta\xi)$ according to the ordinary rule for multiplication of determinants. We thus obtain

$$(abx)(\alpha\beta\xi) = \begin{vmatrix} (a\alpha) & (a\beta) & (a\xi) \\ (b\alpha) & (b\beta) & (b\xi) \\ (x\alpha) & (x\beta) & (x\xi) \end{vmatrix}$$

$=(ab\cdot\overline{a\beta})(x\xi)+(a\beta)(b\xi)(x\alpha)+(a\xi)(b\alpha)(x\beta)-(a\alpha)(b\xi)(x\beta)-(b\beta)(a\xi)(\alpha x).$

Writing

$$s_1 = a\alpha, \qquad s_2 = b\beta,$$

$$\sigma_1 = \alpha a, \qquad \sigma_2 = \beta b,$$

and using the notation of Art. 9 for the linear invariants, we have

$$(17) \qquad \overline{s_1 s_2} = (\overline{s_1 s_2}) + \sigma_1\sigma_2 + \sigma_2\sigma_1 - (s_1)\sigma_2 - (s_2)\sigma_1,$$

where the bar over $s_1 s_2$ is used to signify the operation of forming the intermediate.

13. The intermediate belongs to a type of correspondence that occurs in any number of dimensions. And though we are at present concerned with the collineation in the plane it may be worth the trouble to indicate the general theory.

It is very easy to extend the construction already given for the plane intermediate to the case of higher dimensions. Using the equations (16) we have for the intermediate $\overline{ab}\,\overline{\alpha\beta}$ where $a\alpha$ and $b\beta$ are point collineations in space that to any line $\xi$ corresponds a linear complex with reference to which $\overline{x'y''}$ and $\overline{x''y'}$ are polar lines, $x'$ and $x''$, $y'$ and $y''$, being correspondents with respect to $a\alpha$ and $b\beta$ of two points $x$ and $y$ on $\xi$.

In the same way for the intermediate $\overline{abc}\,\overline{\alpha\beta\gamma}$ we have

$$(18) \qquad \overline{abc}(\alpha\beta\gamma\pi) = \overline{abc}\{(\alpha x)(\beta\gamma\cdot yz) + (\alpha y)(\beta\gamma\cdot zx) + (\alpha z)(\beta\gamma\cdot xy)\}$$

where $x$, $y$, $z$ are three points of the plane $\pi$. If then we take any triangle on the plane $\pi$, transform its points by the collineation $a\alpha$, transform the opposite lines by the collineation $\overline{bc}\,\overline{\beta\gamma}$, and join the corresponding elements, we get a set of three planes intersecting on the correspondent of $\pi$ with respect to the collineation $\overline{abc}\,\overline{\alpha\beta\gamma}$. If $\overline{bc}\,\overline{\beta\gamma}$ is the identical line collineation, or quadratic complex to which every line belongs, the preceding construction is readily translated into a descriptive property of two complete four-points in planes in space.

Proceeding in this way we may construct intermediates in any number of dimensions. Another method was however presented by Kraus * and Muth.† Consider in the first place the plane intermediate written in the form

$$(abx)(\alpha\beta\xi) = 0.$$

This expresses the apolarity of the collineations $\overline{ax}\,\overline{\alpha\xi}$ and $\overline{bx}\,\overline{\beta\xi}$. For on forming the double product we get

$$x(abx)(\alpha\beta\xi)\xi = 0.$$

* "Dissertation" (Giessen), 1886.

† *Math. Annalen*, Bd. 33.

Now $\overline{ax}\,\overline{a\xi}$ and $\overline{bx}\,\overline{\beta\xi}$ may be considered as binary projectivities which give for points on $\xi$ lines joining $x$ to the correspondents with respect to $a\alpha$ and $b\beta$. Then since two binary apolar projectivities give rise to an involution it follows that if we transform the points of a line $\xi$ by the collineations $a\alpha$ and $b\beta$, the correspondent of $\xi$ with respect to $\overline{ab}\,\overline{\alpha\beta}$ is the locus of points from which the transforms appear in involution.

Considering $x$ and $\xi$ as lines in space it follows from the argument of the last paragraph that the collineation $\overline{ab}\,\overline{\alpha\beta}$ in three dimensions gives for a line $\xi$ the complex consisting of lines which joined to the correspondents of points on $\xi$ give pairs of planes belonging to an involution.

In order to interpret the triple intermediate $\overline{abc}\,\overline{\alpha\beta\gamma}$ we need a new invariant. Three plane collineations $a\alpha$, $b\beta$, and $c\gamma$ have in fact a triple product invariant

$$(abc)(\alpha\beta\gamma).$$

When this vanishes the collineations have been called harmonic.* Its vanishing simply expresses that the intermediate of any two of the collineations is apolar to the third. Since we are able to construct the intermediate and to interpret the condition of apolarity this harmonic relation may be supposed known.

The intermediate of three collineations $a\alpha$, $b\beta$, and $c\gamma$ of space may be written in the form

$$(abcx)(\alpha\beta\gamma\pi)=0.$$

This expresses that the three projectivities $\overline{ax}\,\overline{\alpha\pi}$, $\overline{bx}\,\overline{\beta\pi}$, and $\overline{cx}\,\overline{\gamma\pi}$ are harmonic. For on forming the triple product we have

$$x(abcx)(\alpha\beta\gamma\pi)\pi=0.$$

But $\overline{ax}\,\overline{\alpha\pi}$, $\overline{bx}\,\overline{\beta\pi}$, and $\overline{cx}\,\overline{\gamma\pi}$ are ternary correspondences which give for points on $\pi$ lines joining $x$ to their correspondents with respect to $a\alpha$, $b\beta$, and $c\gamma$. Therefore if we construct with respect to $a\alpha$, $b\beta$, and $c\gamma$ the correspondents of points belonging to a plane $\pi$, the correspondent of $\pi$ with respect to $\overline{abc}\,\overline{\alpha\beta\gamma}$ is the locus of points from which those plane systems appear harmonic.

So by a process of continuous induction we may build up intermediates of any degree of complexity. An intermediate of $R$ collineations is reducible to a harmonic invariant of $R$ collineations in $R-1$ dimensions. This may be interpreted as the apolarity of the intermediate of $R-1$ collineations and the remaining one. Thus given the knowledge of apolarity the intermediate of $R$ collineations is reducible to that of $R-1$.

14. A collineation is singular when there is an element whose correspondent is indeterminate. Thus the intermediate $\overline{ab}\,\overline{\alpha\beta}$ in the plane is singular when a line $\xi$ can be found such that

$$\overline{ab}(\alpha\beta\xi)\equiv 0. \tag{19}$$

* J. KRAUS, *Math. Annalen*, Bd. 29, p. 234.

From the construction of the intermediate it is evident that $\xi$ must in this case pass by $a\alpha$ and $b\beta$ into the same line $\eta$. Now (19) is the condition for the correlation

$$a(\alpha\beta\xi)b$$

to be symmetrical, i. e., to be a polarity. But $a(\alpha\beta\xi)b$ sets up a binary correlation on $\eta$ consisting of pairs of points given by $a\alpha$ and $b\beta$ for points of $\xi$. Therefore since every line of the plane cuts $\xi$, it follows that in case of singular intermediate every line of the plane passes by $a\alpha$ and $b\beta$ into a pair of lines apolar to a definite pair of points, i. e., the double points of the binary polarity on the correspondent of $\xi$.

## IV. *Apolarity of collineation and correlation.**

15. A collineation $a\alpha$ and a contragredient correlation $bc$ may be apolar, i. e., may satisfy the condition $ab(c\alpha)=0$. The meaning of this is easily seen. Write them

$$a\alpha = \lambda_1 a_1\alpha_1 + \lambda_2 a_2\alpha_2 + \lambda_3 a_3\alpha_3,$$

$$bc = \mu_1 b_1 c_1 + \mu_2 b_2 c_2 + \mu_3 b_3 c_3,$$

and take $\Delta c = \Delta\alpha$ as reference triangle. The condition of apolarity is then

$$\overline{ab}(c\alpha) = \lambda_1\mu_1\overline{a_1b_1} + \lambda_2\mu_2\overline{a_2b_2} + \lambda_3\mu_3\overline{a_3b_3} = 0.$$

That equation expresses that the triangles $a_i$ and $b_i$ are perspective. Therefore since they are the correspondents through $a\alpha$ and $bc$ of the reference triangle, it follows that *a collineation and an apolar correlation transform any tria s into a pair of perspective triangles.*

Conversely if $a_i$ and $b_i$ are perspective and $a\alpha$ is given, values $\mu_i$ may be found such that (20) holds. Taking those values as the coefficients in $bc$ we have a correlation apolar to $a\alpha$. Therefore if a collineation $a\alpha$ transform the points of a triangle $\alpha$ into a triad perspective to $b_i$, there is a correlation apolar to $a\alpha$ which has $a_i$ and $b_i$ as corresponding pairs. In particular a collineation and correlation are apolar if they transform respectively the points and lines of a triangle into the same triad of points.

The condition that the intermediate $\overline{ab}\,\overline{\alpha\beta}$ of two collineations should be apolar to a correlation $cd$ is

$$\begin{aligned}(abc)\overline{\alpha\beta}\cdot d &= (abc)\{\alpha(\beta d) - \beta(\alpha d)\},\\ &= \overline{bc}(\beta d)\cdot a\alpha + \overline{ac}(\alpha d)\cdot b\beta = 0.\end{aligned}$$

Now $\overline{bc}(\beta d)=0$, and $\overline{ac}(\alpha d)=0$, are the conditions of apolarity of $b\beta$ and $a\alpha$ with $cd$. Hence if two collineations are apolar to a correlation, so is their inter-

* F. ASCHIERI called such correspondences *harmonic*. Compare his article, "Sulle omografie binarie e ternarie," *Rend. del R. Istituto Lombardo*, (2) vol. 24, p. 289.

mediate. By an entirely analogous process it follows that if two correlations are apolar to the same collineation their intermediate is also.

The intermediate of a collineation or correlation with itself is the reciprocal or adjoined form. Hence if a collineation and correlation are apolar the same relation subsists when either or both are replaced by their adjoined forms.

If, for example, two collineations transform the points of a triangle $\alpha$ into perspective triads we have seen that there is a correlation apolar to both collineations which transforms the lines $\alpha_i$ into either of those triads. The preceding paragraph then expresses that, if two collineations transform $\alpha$ into a pair of perspective triangles, the intermediate gives a triangle perspective to both.*

16. A correlation apolar to the identical collineation transforms any triangle into a perspective one and is therefore a polarity. The condition that $bc$ be a polarity is then

$$\overline{bc} = 0. \tag{21}$$

Suppose a collineation $a\alpha$ is apolar to a polarity $bc$. We then have

$$\overline{ab}(\alpha c) = 0,$$

$$\overline{bc} = 0. \tag{22}$$

Let $\xi$ be a fixed line of $a\alpha$ given by the root $\lambda$ of the characteristic equation, i. e., such that

$$(\xi a)\alpha = \lambda\xi.$$

Multiplying by $\xi$ we get from (22)

$$(\alpha c)\{(\xi b)a - (\xi a)b\} = (\alpha c)(\xi b)a - \lambda(\xi c)b = 0,$$

$$(\xi c)b - (\xi b)c = 0.$$

Combining these equations we obtain

$$(\xi c)(b\alpha)a = \lambda(\xi c)b.$$

*For a fixed line of a collineation, an apolar polarity gives a fixed point corresponding to the same root of the characteristic equation.* If the characteristic equation has three distinct roots the fixed triangle is then self-conjugate or polar with respect to any apolar polarity.

From (17) we have for the adjoined form of a collineation $s$,

$$\overline{ss} = (\overline{ss}) + 2\sigma^2 - 2(s)\sigma, \tag{23}$$

where $\sigma$ is the inverse of $s$. Since a polarity is symmetrical, if it is apolar to $s$, it is apolar to $\sigma$. In that case we have also seen that it is apolar to $\overline{ss}$ and

* Cf. MUTH, *Math Annalen*, Bd. 40, p. 98.

identity. Therefore from (23) it follows that if a polarity is apolar to $s$ it is apolar to $\sigma^2$. All collineations that are covariants of $\sigma$ are however expressible linearly in terms of $\sigma^0$, $\sigma$ and $\sigma^2$.* Consequently, *a polarity apolar to a collineation is apolar to all of its covariants.*

If a collineation $s$ transforms the points of a triangle $a$ into a perspective triad, there is a polarity which transforms the lines of $a$ into the same triad. That polarity is obviously apolar to $s$ and to all of its covariants. Further there is a polarity which leaves $a$ fixed and is apolar to $s$. Therefore we have Muth's theorem that if a collineation $s$ transforms a triangle into a perspective one, then all of the covariants of $s$ transform it into triangles perspective to each other and to the original triangle.†

17. For a correlation to be apolar to a collineation requires the identical vanishing of a line and therefore subjects the coefficients of either to three linear conditions. There is not then in general a correlation apolar to each of three given collineations. The condition for such is the vanishing of the determinant of the nine equations expressing the three apolarities. This invariant, the explicit form of which does not concern us, has been called $\Delta$.‡ It is a combinant symmetrical in the coefficients of the three collineations, and of the third degree in each. We will now consider the peculiarities of a system of three collineations for which $\Delta$ vanishes.

In the net

(24) $$s = \rho_1 s_1 + \rho_2 s_2 + \rho_3 s_3$$

there are a single infinity of singular collineations. The singular points lie on a cubic that we may call $C$; the singular lines envelop a cubic that we may call $\Gamma$. It is well known that the adjoined form of a singular collineation consists of the product of singular line and singular point. And we have seen that a correlation apolar to a collineation is apolar to its adjunct. Therefore, *if a correlation is apolar to the collineations $s_1$, $s_2$ and $s_3$, the singular lines and points of* (24) *are correspondents in that correlation and consequently the cubics $\Gamma$ and $C$ are reciprocal through it.*

With a point of $C$ is associated in two ways a line of $\Gamma$. In the first place the point $a$ appears as singular point in a definite collineation of (24) which has a singular line $\beta$. Secondly, it is transformed by those collineations into the points of a definite line $\gamma$.

To say that a correlation is apolar to each of three collineations amounts to saying that those collineations operating on the inverse of that correlation give polarities. Such a transformation of the collineations does not affect the cubic $\Gamma$ which is therefore the Cayleyan of the three polarities. We saw above how-

* CLEBSCH, *Vorlesungen über Geometrie*, Bd. 1, p. 991.

† MUTH, loc. cit., p. 97.

‡ ROSANES, *Crelle*, Bd. 95, p. 254.

ever that the line $\beta$ passes by the inverse of the apolar correlation into the point $a$. Therefore $\beta$ passes by the three polarities into points of $\gamma$ and consequently $\beta$ and $\gamma$ are corresponding lines of $\Gamma$.

Conversely, suppose with each point of $C$ are associated a pair of corresponding lines of $\Gamma$. Two collineations of (24) whose common polar triangles are not singular may be written

$$s_1 = \lambda_1 b_1 a_1 + \lambda_2 b_2 a_2 + \lambda_3 b_3 a_3,$$

$$s_2 = \mu_1 b_1 a_1 + \mu_2 b_2 a_2 + \mu_3 b_3 a_3.$$

The lines $\beta_i$ are singular in the collineations whose singular points are $a_i$. The points $a_1$, $a_2$, $a_3$ pass by the net of collineations respectively into points of

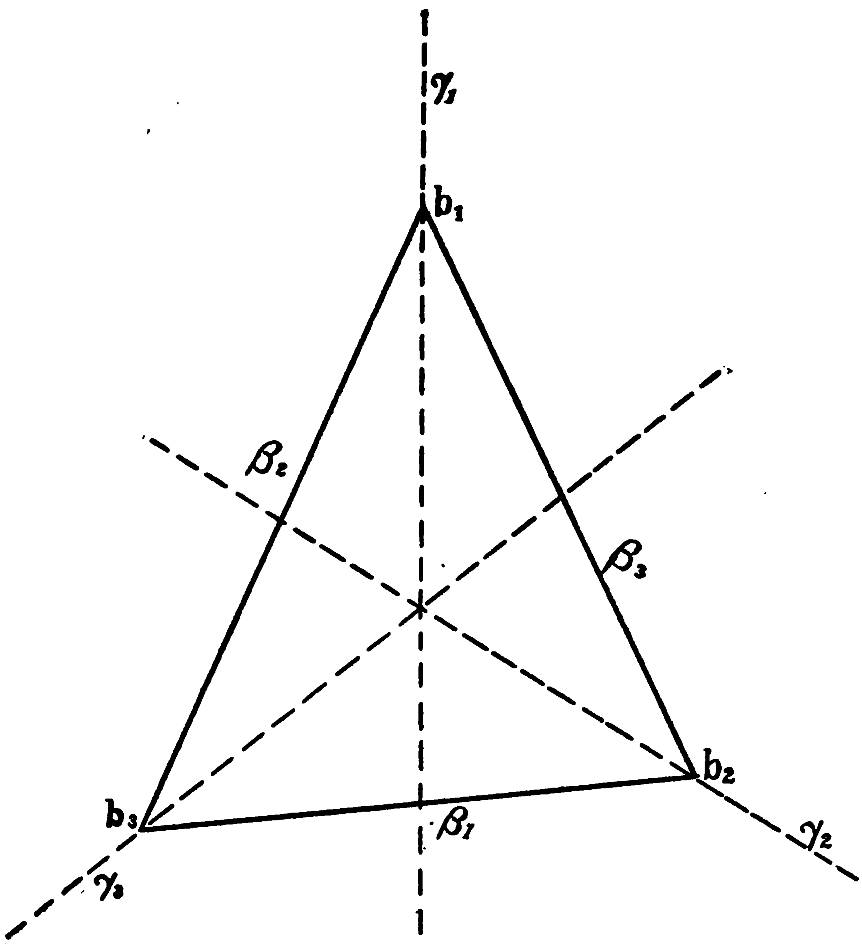


Fig. 2.

$\gamma_1, \gamma_2, \gamma_3$ where $\gamma_i$ is incident to $b_i$. By supposition $\beta_1\gamma_1$, $\beta_2\gamma_2$, and $\beta_3\gamma_3$ are three pairs of corresponding lines with respect to the curve. If we should start with $b_1$ we could construct a complete four-point, all of whose lines touch the curve. It would contain those three pairs of lines as diagonal pairs and therefore $\gamma_1, \gamma_2, \gamma_3$ pass through a point (Fig. 2). The triad $a_i$ then passes by two of the collineations into the triad $b_i$ and by the third into one perspective to $b_i$. According to Art. 15, the three collineations therefore have a common apolar correlation.

*The vanishing of the invariant $\Delta$ is the necessary and sufficient condition that with any point of $C$ should be associated (in the above way) a pair of corresponding lines of $\Gamma$, and, dually, with any line of $\Gamma$ should be associated a pair of corresponding points of $C$.*

The similarity of a set of collineations for which $\Delta$ vanishes to a net of polarities is noticeable. It is due to the fact that a set of polarities are apolar to the identical collineation, or, what amounts to the same thing, that a net of collineations with vanishing $\Delta$ may be transformed into a net of polarities in such a way as to have either the initial or the resultant field invariant.

## PART III. THE ALTERNANT.

### I. *Introduction.*

16. In contrast with the symmetrical forms just considered are a series of combinants that we will call *alternants*. The alternant of $n$ collineations $s_i$ is defined by the equation

$$[s_1 \cdots s_n] = \begin{vmatrix} s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_1 & s_2 & \cdots & s_n \end{vmatrix},$$

where the determinant is supposed to be developed in the order of its columns, i. e., in each term of the development the first letter is taken from the first row, the second from the second row, etc. This determinant is readily seen to follow the ordinary rules so far as its rows are concerned. If, for example, a linear relation exists between the collineations $s_1, \cdots, s_n$, the alternant is zero. Using the ordinary rule of signs, the determinant may be developed as the sum of products by their minors of determinants of $r$th order in the first $r$ rows. The alternant cannot, however, in general be developed in terms of its columns.

Obviously, there will be a marked difference according as the order of the alternant is odd or even. If the order $n$ is even, for every term of the form $s_i \cdots s_j s_k$ will be a term $-s_k s_i \cdots s_j$ where the intervening letters in both are

the same. The linear invariant is therefore

$$\sum \{ (s_i \cdots s_j s_k) - (s_k s_i \cdots s_j) \} .$$

*The alternant of an even number of collineations is normal.*

Again if $n$ is even the alternant may be written in the form

(26) $$\sum (P s_i Q - Q s_i P)$$

where $P$ and $Q$ are products not containing $s_i$. This is harmonic with $s_i$* since

$$\sum \{ (s_i P s_i Q) - (s_i Q s_i P) \} = 0 .$$

*The alternant of an even number of collineations is harmonic with each of them.*

Finally from (26) placing $s_i$ to be identity we see that *the alternant of an even number of collineations vanishes when one of those collineations is identity.*

In case of an odd alternant, since the members of a cyclic group are all of the same sign, we have

(27) $$([s_1 \cdots s_n]) = n(s_1 [s_2 \cdots s_n]) .$$

*If the alternant of an odd number of collineations is normal, each collineation is harmonic with the alternant of the remaining* $n-1$.

Write the alternant in the form

$$[s_1 \cdots s_n] = \sum s_i S_i ,$$

where $S_i$ is the minor of $s_i$ in the first row of the alternant. If $n$ is odd and one of the collineations $s_1$ is identity, since the first minors are even, all those containing identity vanish and the alternant takes the form

$$[s_1 \cdots s_n] = s_0 (s_2 \cdots s_n] = [s_2 \cdots s_n] .$$

*The alternant of an odd number of collineations containing identity is equal to the alternant of the remaining* $n-1$.

## II. *The binary alternant.*

17. The alternant $[s_1 s_2]$ of two collineations is normal and harmonic with each of the collineations. A binary normal collineation, or polarity, is apolar to a collineation $s$ only when the fixed points of $s$ form a pair in that polarity. Therefore $[s_1 s_2]$ is the polarity determined by the common harmonic pair of the fixed points of $s_1$ and $s_2$.†

Consider in the next place the triple alternant

$$[s_1 s_2 s_3] = \begin{vmatrix} s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 \end{vmatrix} .$$

* See Art. 9, footnote.

† STUDY, "Benäre Formen."

It is normal when $s_1$ is harmonic with $[s_2 s_3]$. Therefore the vanishing of the linear invariant of $[s_1 s_2 s_3]$ expresses the condition that the fixed points of $s_1$, $s_2$, and $s_3$ should lie in an involution.

The condition that a collineation

$$\rho_1 s_1 + \rho_2 s_2 + \rho_3 s_3 \tag{28}$$

be harmonic with $[s_1 s_2 s_3]$ is

$$0 = \rho_1 \{(s_1^2 s_2 s_3) - (s_1^2 s_3 s_2)\} + \rho_2\{(s_2^2 s_3 s_1) - (s_2^2 s_1 s_3)\} + \rho_3\{(s_3^2 s_1 s_2) - (s_3^2 s_2 s_1)\}.$$

Making use of the identity $s^2 = is - \Delta$, this becomes

$$0 = \{\rho_1(s_1) + \rho_2(s_2) + \rho_3(s_3)\}\{(s_1 s_2 s_3) - (s_3 s_2 s_1)\}.$$

The alternant is therefore harmonic with all of the normal collineations of the pencil $\rho_1 s_1 + \rho_2 s_2 + \rho_3 s_3$. *The fixed points of* $[s_1 s_2 s_3]$ *give the collineations of* (28) *whose dyadics are squares.*

It is readily seen that the collineation

$$s = [s_1 s_2 s_3] - 2\{(s_1 s_2 s_3) - (s_2 s_3 s_1)\}$$

is harmonic with $s_1$, $s_2$, and $s_3$. When $s$ is identity $[s_1 s_2 s_3]$ is identity and conversely. *The condition that* $s_1$, $s_2$, *and* $s_3$ *should be polarities is that the alternant* $[s_1 s_2 s_3]$ *should coincide with the identical collineation.*

When $[s_1 s_2 s_3]$ vanishes, the invariant $(s_1 s_2 s_3) - (s_3 s_2 s_1)$ also vanishes and hence $s$ is zero. The three collineations $s_1$, $s_2$, $s_3$ have no definite apolar collineation and therefore satisfy a linear identity. The vanishing of $[s_1 s_2 s_3]$ is the condition for the three collineations to satisfy a linear identity.

From any four linearly independent binary collineations any other may be linearly derived. In particular to such a set of four belongs the identical collineation. Since an alternant of even order containing identity is zero, it follows that the binary alternant of fourth order vanishes identically. Therefore with the alternant $(s_1 s_2 s_3)$ just considered the discussion in the binary domain comes to an end.

### III. *The alternant of two ternary collineations.*

18. We shall usually write the collineations in the form

$$s_1 = a\alpha,$$

$$s_2 = b\beta.$$

The alternant is then

$$[s_1 s_2] = s_1 s_2 - s_2 s_1 = (ab)a\beta - (\beta a)b\alpha.$$

The covariants of a collineation $s$ are linear functions of $s_0$, $s$, and $s^2$, where $s_0$ is the identical collineation. Since each of these is commutative with $s$ it

follows that *the alternant of a collineation and any of its covariants vanishes identically.*

The invariant relations of the alternant and covariants of $s_1$ and $s_2$ may be summed up in two general theorems.

(i). *The alternant is apolar to all the covariants of $s_1$ or $s_2$.*

For let $c\gamma$ be any covariant of $s_1$. The condition of apolarity with the alternant is

$$\begin{aligned} 0 &= (ab)(a\gamma)(\beta c) - (\beta a)(b\gamma)(ac) \\ &= b\beta \cdot [(a\gamma)ac - (ac)\gamma a], \end{aligned}$$

where the dot is used to represent the process of forming the double product, or bilinear invariant. Since the expression in brackets is the alternant of $a\alpha$ and a covariant $c\gamma$, the function vanishes as was required.

(ii). *The alternant is apolar to the intermediate of one of the collineations and any covariant of the other.*

For let $c\gamma$ again be a covariant of $a\alpha$. The intermediate of this with $b\beta$ is

$$\overline{bc}\,\overline{\beta\gamma}.$$

The condition of apolarity with the alternant is

$$0 = (ab')(abc)(\beta'\beta\gamma) - (\beta'a)(b'bc)(a\beta\gamma),$$

where $b'\beta'$ is a new symbol for $b\beta$. Interchanging $b\beta$ and $b'\beta'$ and adding, the last expression becomes

$$\tfrac{1}{2}\{(\beta'\beta\gamma)(\overline{b'b}\,\overline{a\cdot ca}) - (b'bc)(\overline{\beta'\beta}\,\overline{a\cdot\gamma a})\} = \tfrac{1}{2}\overline{\beta'\beta}\,\overline{b'b}\cdot[\gamma\overline{a\cdot ca} - \overline{a}\cdot\overline{\gamma ac}].$$

The expression in brackets expands into

$$(aa)\gamma c - (ac)\gamma a - (aa)\gamma c + (a\gamma)ac = (a\gamma)ac - (ac)\gamma a,$$

which is zero since it is the alternant of $a\alpha$ and a covariant.

19. Since all covariants of $s$ are expressible linearly in terms of $s_0$, $s$, $s^2$, the alternant is found to be apolar to the eight collineations

$$\sigma_0,\ \sigma_1,\ \sigma_1^2,\ \sigma_2,\ \sigma_2^2,\ \overline{s_1 s_2},\ \overline{s_1^2 s_2},\ \overline{s_1 s_2^2}, \tag{29}$$

where, as in Art. 9, $\overline{s_1 s_2}$ is the intermediate of $s_1$ and $s_2$ and $\sigma$ is the same connex as $s$ but considered reciprocally. If the eight collineations are linearly independent, the eight apolarity conditions are sufficient uniquely to determine the alternant. It is our purpose in the next place to see whether or when such is the case.

By the formula (17) we have

$$\overline{s_1 s_2^2} = (\overline{s_1^2 s_2^2}) + \sigma_1^2\sigma_2^2 + \sigma_2^2\sigma_1^2 - (s_1^2)\sigma_2^2 - (s_2^2)\sigma_1^2.$$

Since the alternant $[s_1 s_2]$ is apolar to $\sigma_0$, $\sigma_1^2$ and $\sigma_2^2$, we see that the condition of apolarity of $[s_1 s_2]$ and $\overline{s_1^2 s_2^2}$ is the vanishing of the linear invariant of the collineation

$$(s_1 s_2 - s_2 s_1)(s_1^2 s_2^2 + s_2^2 s_1^2).$$

Hence by direct expansion of this last named invariant we obtain

$$\begin{aligned}(30)\qquad [s_1 s_2] \cdot \overline{s_1^2 s_2^2} &= (s_1 s_2 s_1^2 s_2^2) + (s_1 s_2^3 s_1^2) - (s_2 s_1^3 s_2^2) - (s_2 s_1 s_2^2 s_1^2) \\ &= (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2),\end{aligned}$$

since by Art. 9 $(s_1 s_2^3 s_1^2)$ and $(s_2 s_1^3 s_2^2)$ are equal, both passing by a cyclic permutation into $(s_1^3 s_2^3)$.

Again we have

$$\overline{s_1^2 s_2} = (\overline{s_1^2 s_2}) + \sigma_1^2 \sigma_2 + \sigma_2 \sigma_1^2 - (s_1^2)\sigma_2 - (s_2)\sigma_1^2.$$

Since $[s_2^2 s_1]$ is apolar to $\sigma_0$, $\sigma_2$ and $\sigma_1^2$, by the same argument as before, the apolarity condition of $[s_2^2 s_1]$ and $\overline{s_1^2 s_2}$ is found to be the linear invariant of the collineation

$$(s_2^2 s_1 - s_1 s_2^2)(s_1^2 s_2 + s_2 s_1^2).$$

Expanding and making use of the cyclic permutation we then obtain

$$\begin{aligned}(31)\qquad [s_2^2 s_1] \cdot \overline{s_1 s_2^2} &= (s_2^2 s_1^3 s_2) + (s_2^2 s_1 s_2 s_1^2) - (s_1 s_2^2 s_1^2 s_2) - (s_1 s_2^3 s_1^2) \\ &= (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2).\end{aligned}$$

In like manner, making use of the identity

$$\overline{s_1 s_2} = (\overline{s_1 s_2}) + \sigma_1 \sigma_2 + \sigma_2 \sigma_1 - (s_1)\sigma_2 - (s_2)\sigma_1,$$

we obtain the apolarity condition of $[s_1^2 s_2^2]$ and $\overline{s_1 s_2}$ as the linear invariant of

$$(s_1^2 s_2^2 - s_2^2 s_1^2)(s_1 s_2 + s_2 s_1)$$

under the form

$$(32)\qquad [s_1^2 s_2^2] \cdot \overline{s_1 s_2} = (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2).$$

Since $[s_1 s_2]$ is a normal collineation the adjoined form is given by (17) as

$$\overline{[s_1 s_2][s_1 s_2]} = (\overline{[s_1 s_2][s_1 s_2]}) + 2[\sigma_1 \sigma_2]^2.$$

The discriminant of $[s_1 s_2]$ is then

$$\begin{aligned}\Delta_{11} &= \tfrac{1}{6}[s_1 s_2] \cdot \overline{[s_1 s_2][s_1 s_2]} = \tfrac{1}{3}[s_1 s_2] \cdot [\sigma_1 \sigma_2]^2 = \tfrac{1}{3}([s_1 s_2]^3) \\ (33)\qquad &= \tfrac{1}{3}\{(s_1 s_2 s_1 s_2 s_1 s_2) - (s_1 s_2 s_1 s_2^2 s_1) - (s_1 s_2^2 s_1^2 s_2) - (s_2 s_1^2 s_2 s_1 s_2) \\ &\quad - (s_2 s_1 s_2 s_1 s_2 s_1) + (s_2 s_1 s_2 s_1^2 s_2) + (s_2 s_1^2 s_2^2 s_1) + (s_1 s_2^2 s_1 s_2 s_1)\} \\ &= (s_1 s_2 s_1^2 s_2^2) - (s_2 s_1 s_2^2 s_1^2),\end{aligned}$$

in which the notation $\Delta_{11}$ is used to show that it is the discriminant of an alternant involving $s_1$ and $s_2$ each to the first degree.

A comparison of (30), (31), and (32) with (33) shows that the various invariants there considered are equal to each other and to the discriminant of $[s_1 s_2]$.

Now suppose that $\Delta_{11}$ does not vanish and that there exists a linear relation of the form

$$(34)\quad \rho_0\sigma_0 + \rho_1\sigma_1 + \rho_2\sigma_1^2 + \rho_3\sigma_2 + \rho_4\sigma_2^2 + \rho_5\overline{s_1 s_2} + \rho_6\overline{s_1^2 s_2} + \rho_7\overline{s_1 s_2^2} + \rho_8\overline{s_1^2 s_2^2} = 0.$$

Since according to (29), $[s_1 s_2]$ is apolar to the first eight collineations in that sequence but by (30) and (33) is not apolar to the last, it follows that $\rho_8$ is zero. Likewise, operating in turn with $[s_1^2 s_2]$, $[s_2^2 s_1]$ and $[s_1^2 s_2^2]$, we find that $\rho_7 = \rho_6 = \rho_5 = 0$. Hence the relation (34) must be of the form

$$(35)\qquad \rho_0\sigma_0 + \rho_1\sigma_1 + \rho_2\sigma_1^2 + \rho_3\sigma_2 + \rho_4\sigma_2^2 = 0$$

Replacing $\sigma_i$ by $s_i$ and forming the alternant with $s$, we obtain

$$(36)\qquad \rho_3[s_1 s_2] + \rho_4[s_1 s_2^2] = 0.$$

Forming the bilinear invariant with $\overline{s_1^2 s_2^2}$, this gives

$$\rho_3\Delta_{11} = 0.$$

Hence $\rho_3 = 0$. Likewise on operating with $\overline{s_1^2 s_2}$, it is seen that $\rho_4 = 0$. Similarly $\rho_2 = \rho_1 = \rho_0 = 0$.

Hence if the discriminant $\Delta_{11}$ is different from zero no linear relation of the type (34) can exist. If however the discriminant is zero, since the nine collineations are apolar to $[s_1 s_2]$, they must satisfy a linear relation. Therefore, *the vanishing of the discriminant of the alternant is the necessary and sufficient condition for the existence of a linear relation of the type* (34).

Since in the usual case the discriminant of the alternant is not zero, it follows that in general the apolarity conditions of (29) are independent and give an invariant determination of the alternant. When the discriminant is zero, all the collineations of the net

$$(37)\qquad \lambda_1[s_1 s_2] + \lambda_2[s_1^2 s_2] + \lambda_3[s_1 s_2^2] + \lambda_4[s_1^2 s_2^2]$$

satisfy those conditions and the determination is not unique.

20. When the discriminant vanishes there is always a collineation of (37) that vanishes identically, i. e., the four alternants satisfy a linear relation. For since all the collineations of (37) are apolar to all those in (34) it follows that the two sets must contain four linear relations. If one of these belongs to (37) the point at issue is settled. If not there must be four equations of the type (34). Either one of those is of the form (35) and the collineation in question is (36), or it is possible to solve for one of the intermediates and so obtain

an equation

$$\overline{s_1 s_2} = \lambda_0 \sigma_0 + \lambda_1 \sigma_1 + \lambda_2 \sigma_1^2 + \lambda_3 \sigma_2 + \lambda_4 \sigma_2^2 .$$

Developing $\overline{s_1 s_2}$ by (17), inverting, and forming the alternant with $s_1$, gives

$$[s_1^2 s_2] - (s_1)[s_1 s_2] = \lambda_3 [s_1 s_2] + \lambda_4 [s_1 s_2^2]$$

which is the relation desired.

Suppose conversely that the four alternants satisfy a linear relation

$$\lambda_1 [s_1 s_2] + \lambda_2 [s_1^2 s_2] + \lambda_3 [s_1 s_2^2] + \lambda_4 [s_1^2 s_2^2] = 0 .$$

In this equation there must be at least one coefficient, for instance $\lambda_1$, that is different from zero. Operating on the equation with $\overline{s_1^2 s_2^2}$ we see that $\Delta_{11}$ must then vanish. Therefore, the vanishing of the discriminant is the necessary and sufficient condition for a linear relation between the four alternants.

21. The symmetry resulting when $\Delta_{11}$ vanishes suggests that it is an invariant common to the four alternants. In order to prove that such is the case, take in the first instance the alternant $[s_1^2 s_2]$. According to (33), the discriminant has the form

$$\Delta_{21} = (s_1^2 s_2 s_1^4 s_2^2) - (s_2 s_1^2 s_2^2 s_1^4) .$$

From the characteristic equation for $s_1$ we have

$$s_1^4 = \lambda_0 + \lambda_1 s_1 + \lambda_2 s_1^2 .$$

Substituting this value for $s_1^4$ we have

$$\Delta_{21} = \lambda_1 \{ (s_1^2 s_2 s_1 s_2^2) - (s_2 s_1^2 s_2^2 s_1) \} = -\lambda_1 \Delta_{11} . \tag{38}$$

Making use of

$$s_2^4 = \mu_0 + \mu_1 s_2 + \mu_2 s_2^2$$

and following out the same argument we obtain the discriminants of $[s_1 s_2^2]$ and $[s_1^2 s_2^2]$ in the form

$$\begin{cases} \Delta_{12} = -\mu_1 \Delta_{11} , \\ \Delta_{22} = \lambda_1 \mu_1 \Delta_{11} . \end{cases} \tag{39}$$

If $\lambda_1$ is zero the characteristic equation for $s_1^2$ is

$$\{ s_1^2 \}^2 = \lambda_0 + \lambda_2 s_1^2 .$$

The equation is quadratic and hence the collineation is a perspectivity. Therefore $\lambda_1$ and $\mu_1$ are respectively the invariants whose vanishing expresses that $s_1^2$ and $s_2^2$ are perspectivities. From (38) and (39) we see then that *the alternant of a perspectivity and any collineation is singular.*

If $\lambda_1$ and $\mu_1$ are not zero $\Delta_{11}$ is a combinant of the two systems $\lambda_0 s_0 + \lambda_1 s_1 + \lambda_2 s_1^2$ and $\mu_0 s_0 + \mu_1 s_2 + \mu_2 s_2^2$. It must then express a property of the fixed triangles of those systems. What that property is we shall see later.

22. The alternant $[s_1 s_2]$ is a combinant of the net

$$\lambda s_0 + \mu s_1 + \nu s_2. \tag{40}$$

In forming it any two independent collineations may then be chosen. Let $s$ be a singular collineation, $x$ its singular point, and $\xi$ its singular line. The alternant may be written

$$ss_1 - s_1 s$$

where $s_1$ is some other collineation of the net. Since $sx$ is zero the correspondent of $x$ with respect to the alternant is

$$x' = ss_1 x.$$

Now $s$ transforms every point (and in particular $s_1 x$) into a point on $\xi$. Therefore the alternant transforms the singular point of any collineation of (40) into a point of the associated singular line. For varying $\lambda$, the singular points and lines obtained are the fixed points and associated fixed lines in the collineation $\mu s_1 + \nu s_2$. Therefore *the fixed triangles of all the collineations of the pencil* $\mu s_1 + \nu s_2$ *are Pasch triangles of the alternant.*

The significance of the apolarity relations satisfied by the alternant is here suggested. In fact two collineations (one in points, the other in lines) that send a triangle into a pair of incident triangle are apolar. Now if $s$ is any collineation in (40) the triangle that $s$ leaves fixed is sent by the alternant into an inscribed triangle. Therefore the alternant is apolar to $s$ and to all of its covariants.

Two collineations have in general a common pair of polar triangles. In terms of these they may be written

$$s_1 = \lambda_1 b_1 \alpha_1 + \lambda_2 b_2 \alpha_3 + \lambda_3 b_3 \alpha_3,$$
$$s_2 = \mu_1 b_1 \alpha_1 + \mu_2 b_2 \alpha_2 + \mu_3 b_3 \alpha_3.$$

Their adjoined forms and intermediate are respectively

$$\overline{s_1 s_1} = 2\{\lambda_1\lambda_2\beta_3 a_3 + \lambda_2\lambda_3\beta_1 a_1 + \lambda_3\lambda_1\beta_2 a_2\},$$
$$\overline{s_2 s_2} = 2\{\mu_1\mu_2\beta_3 a_3 + \mu_2\mu_3\beta_1 a_1 + \mu_3\mu_1\beta_2 a_2\},$$
$$\overline{s_1 s_2} = (\lambda_1\mu_2 + \lambda_2\mu_1)\beta_3 a_3 + (\lambda_2\mu_3 + \lambda_3\mu_2)\beta_1 a_1 + (\lambda_3\mu_1 + \lambda_1\mu_3)\beta_2 a_2.$$

Hence $\overline{s_1 s_1}$, $\overline{s_2 s_2}$, and $\overline{s_1 s_2}$ have a common pair of polar triangles, i. e., the common pair of $s_1$ and $s_2$ considered contragrediently. Since $[s_1 s_2]$ is apolar to each of those collineations, according to Art. 10, it transforms the triad $a_i$ into one incident to $\beta_i$. Taking any two collineations and the associated intermediate belonging to (29), we obtain a pair of polar triangles such that the points of the first pass by the alternant into a triad incident to the second. These relations are therefore the geometric equivalent of the eight apolarity conditions.

23. In Arts. 19–21 the entire theory seemed to hinge on the vanishing or non-vanishing of the discriminant of the alternant. It is our purpose in the next place to consider the geometrical interpretation of that invariant.

For that purpose suppose a polarity $c^2$ to be apolar to $s_1$ and $s_2$. Designating the collineations respectively by $a\alpha$ and $b\beta$, the conditions required are

$$\overline{ac}(\alpha c) = \overline{bc}(\beta c) = 0. \tag{41}$$

There are six equations in all, three of them linear in the coefficients of $a\alpha$, three linear in the coefficients of $b\beta$, and all linear in the coefficients of $c^2$. Therefore if we eliminate the coefficients of $c^2$ from these equations we get an invariant of the third degree in the coefficients of $a\alpha$ and $b\beta$ whose vanishing is the necessary and sufficient condition for the equations (41).

Now we have seen that a polarity apolar to a collineation is apolar to all of its covariants, and that a correlation apolar to two collineations is apolar to their intermediate. Therefore $c^2$ is apolar to the following nine collineations,

$$\sigma_0,\ \sigma_1,\ \sigma_1^2,\ \sigma_2,\ \sigma_2^2,\ \overline{s_1 s_2},\ \overline{s_1^2 s_2},\ \overline{s_1 s_2^2},\ \overline{s_1^2 s_2^2}. \tag{42}$$

Since the nine collineations are apolar to the same polarity they must satisfy a linear relation, in fact, must satisfy three linear relations. Therefore according to Art. 19, the discriminant of the alternant is zero. But the discriminant is of the third degree in the coefficients of $a\alpha$ and $b\beta$. Therefore, the vanishing of the discriminant of the alternant is the necessary and sufficient conditions for two collineations to have a common apolar polarity.

If a polarity is apolar to a collineation there are two cases to be considered according as the polarity is singular or is not.

If the polarity is singular it either consists of the square of a point or its adjoined form consists of the square of a line. In both cases if the polarity is apolar to a collineation its fixed triangle contains the double element. Hence if a singular polarity is apolar to each of two collineations their fixed triangles have an element in common. In that case all the collineations of (42) have a fixed point or line in common.

In general, however, if a polarity is apolar to a collineation, the fixed triangle (or, a fixed triangle) of the collineation is self-conjugate with respect to the polarity. If two triangles are self-conjugate with respect to the same polarity they lie on a conic. And, conversely, if two fixed triangles lie on a conic, they are self-conjugate with respect to a polarity, which is consequntly apolar to their associated collineations. Therefore we see that *the vanishing of the discriminant of the alternant is the necessary and sufficient condition for any two collineations formed linearly from those of* (42) *to have fixed triangles lying on a conic.*

24. As we have seen the conditions for a polarity $c^2$ apolar to $a\alpha$ and $b\beta$ are

$$(\alpha c)\overline{ac} = (\beta c)\overline{bc} = 0.$$

Multiplying these equations by $b\beta$ and $a\alpha$ and writing in the variables to avoid confusion we get

$$(\alpha c)\{(\beta c)(a\xi)-(\beta a)(c\xi)\}(b\eta)\equiv 0,$$

$$(\beta c)\{(\alpha c)(b\xi)-(\alpha b)(c\xi)\}(a\eta)\equiv 0.$$

Placing $\xi$ and $\eta$ equal and subtracting, these equations give

(43) $$\{(\alpha b)(\beta c)(a\xi)-(\beta a)(\alpha c)(b\xi)\}(c\xi)\equiv 0.$$

Comparing this with the general form

$$\{(\alpha b)a\beta-(\beta a)b\alpha\}c^2$$

we see that in the present case the product of the alternant with $c^2$ is a correlation having no particular coincidence conic, i. e., a nullsystem or line.

Suppose for the moment we designate the alternant by $d\delta$. The equation (43) is then

(44) $$(d\xi)(\delta c)(c\xi)=0,$$

where $\xi$ is any line whatever. Let $\eta$ be a fixed line of the alternant. We then have

$$(d\eta)\delta=\lambda\eta,$$

which substituted in the preceding equation gives

$$(d\eta)(\delta c)(c\eta)=\lambda(\eta c)^2=0.$$

Hence the fixed lines of the alternant touch the conic of $c^2$.

If again we follow out the argument of this article with $\gamma^2$, the adjoined form of $c^2$, we obtain as correlative to (44)

(45) $$(\gamma d)(x\delta)(x\gamma)=0$$

where $x$ is any point whatever. Taking $x$ as a fixed point of the alternant, it follows as before that the fixed lines of the alternant lie on $\gamma^2$.

Now, we have found that when $c^2$ is apolar to $s_1$ and $s_2$, it is apolar to a set of nine covariants. Therefore, *all the alternants that can be formed of those covariants are singular, all their fixed lines are tangent to* $c^2$ *and all their fixed points lie on* $\gamma^2$.

It was shown by Study that the fixed points of the alternant of two binary collineations consist of the common harmonic pair of the fixed points of those collineations. The fixed triangles of two ternary collineations are not in general polar with respect to the same conic. If such however is the case, we have just seen that the fixed lines of the alternant are tangent to and the fixed points lie on that conic.

## VITA.

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